

Lecture 9:

14 Sep 2016

(1)

→ Prop: Let f, g be holomorphic at $z_0 \in \mathbb{C}$, and let $A \in \mathbb{C}$.

Then:

- $(f+g)'(z_0) = f'(z_0) + g'(z_0)$.
- $(Af)'(z_0) = A \cdot f'(z_0)$.
- $(f \cdot g)'(z_0) = f'(z_0) \cdot g(z_0) + f(z_0) \cdot g'(z_0)$.
- $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0) \cdot g(z_0) - f(z_0) \cdot g'(z_0)}{g^2(z_0)}$, when $g(z_0) \neq 0$.
- $(f \circ g)'(z_0) = f'(g(z_0)) \cdot g'(z_0)$, as long as $f'(g(z_0))$ exists.

Proof: Either with the definition of the derivative at z_0 ,

or by verifying the Cauchy-Riemann conditions and using the usual rules of partial differentiation in \mathbb{R}

(together with $f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \cdot \frac{\partial v}{\partial x}(x_0, y_0)$)

(2)

→ Examples:

- $(z^n)' = n \cdot z^{n-1}$, $\forall n \in \mathbb{N}$ (use product rule).
- $\left(\frac{1}{z}\right)' = -\frac{1}{z^2}$, $\forall z \neq 0$.
- $\left(\frac{1}{z^n}\right)' = -\frac{1}{(z^n)^2} \cdot (z^n)'$, $\forall n \in \mathbb{N}, \forall z \neq 0$ (just like for $z \in \mathbb{R}$).

→ Def: $f: \mathbb{C} \rightarrow \mathbb{C}$ is called analytic at $z_0 \in \mathbb{C}$

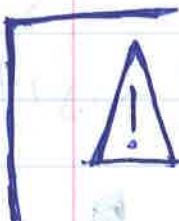
↓
or maybe
defined on
a subset of \mathbb{C}

if f can be written as
a power series around z_0 , i.e.:

if $f(z) = \sum_{k=0}^{+\infty} a_k (z-z_0)^k$, $\forall z \in D(z_0, r)$,

for some $r > 0$.

(it doesn't matter which).



We have discussed that real functions don't have to be analytic, even if they are smooth. However, we will show that, if f is holomorphic at $z_0 \in \mathbb{C}$ (i.e. differentiable once at z_0 in D), then f analytic at z_0 !

(3)

And, in particular, f will be infinitely many times differentiable at z_0 . On the way, some very useful results will come up.

→ Def: Let $f: \underbrace{[a,b]}_{\text{in } \mathbb{R}} \rightarrow \mathbb{C}$ continuous.

We define

$$\int_a^b \underbrace{f(t) dt}_{\in \mathbb{C}} := \int_a^b \operatorname{Re}(f(t)) dt + i \cdot \int_a^b \operatorname{Im}(f(t)) dt.$$

this means that, when I draw the values $f(x)$ on the complex plane, as x runs from a to b , I don't lift my pencil from the paper!



ex: Let $f(t) = \sin t + it$, $t \in [0, 2\pi]$. Then:

$$\int_0^{2\pi} f(t) dt = \int_0^{2\pi} \sin t dt + i \cdot \int_0^{2\pi} t dt = 0 + i \cdot \left[\frac{t^2}{2} \right]_0^{2\pi} = 2 \cdot \pi^2.$$

→ Prop: Let $f, g: [a,b] \rightarrow \mathbb{C}$ continuous, $\lambda \in \mathbb{C}$. Then:

- $\int_a^b (\underbrace{f(t) + g(t)}_{\in \mathbb{C}}) dt = \int_a^b f(t) dt + \int_a^b g(t) dt.$
- $\int_a^b \underbrace{\lambda \cdot f(t)}_{\in \mathbb{C}} dt = \lambda \cdot \int_a^b f(t) dt.$

(4)

- If $h: [a, b] \rightarrow \mathbb{C}$ is such that $h'(t) = f(t) \quad \forall t \in [a, b]$,

then $\int_a^b f(t) dt = [h(t)]_a^b (= h(b) - h(a))$.

Proof: We just split in real and imaginary parts, and use the definition for the integral.

For the last bullet, we just use that

$$h'(t) = (\operatorname{Re} h(t))' + i \cdot (\operatorname{Im} h(t))'$$

as $h'(t) = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon \in \mathbb{R}}} \frac{h(t+\epsilon) - h(t)}{\epsilon}$. ■



$$\begin{aligned} (e^{it})' &= (\cos t + i \sin t)' = (\cos t)' + i \cdot (\sin t)' = \\ &= -\sin t + i \cos t = \\ &= i \cdot (\cos t + i \sin t) = i \cdot e^{it}. \end{aligned}$$

One could see this also by the chain rule for $z=it$, but we haven't shown that e^z is holomorphic yet.

↓
show it with
the Cauchy-Riemann conditions!

→ Def: A curve γ is a continuous function

$$\gamma: [a, b] \rightarrow \mathbb{C}$$



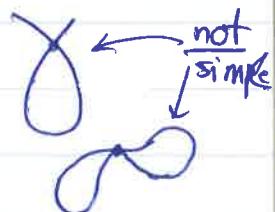
⚠ We denote by γ^* the image of γ in \mathbb{C} ,
i.e. the picture we see here. Beware: γ

is not the same as γ^* ! γ contains all

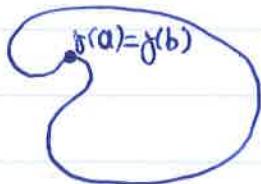
the information about how we parametrised
 γ^* (i.e., it tells us that we move from
 $\gamma(a)$ to $\gamma(b)$, and not from $\gamma(b)$ to $\gamma(a)$;
it also tells us how we move from
 $\gamma(a)$ to $\gamma(b)$).

→ Def: A curve $\gamma: [a, b] \rightarrow \mathbb{C}$ is

- simple, if γ^* doesn't cross itself.



- closed, if $\gamma(a) = \gamma(b)$



(i.e., in γ^* we end up to
the point we started from:
 γ^* is a loop).

(6)

→ Def: Let $\gamma: [a, b] \rightarrow \mathbb{C}$ be a simple, closed curve.

Then, it is known that

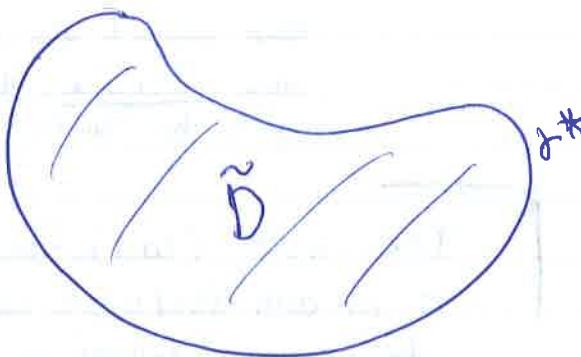
γ^* splits \mathbb{C} in two sets, each of which is path-connected (i.e., $\mathbb{C} \setminus \gamma^* = A \cup B$, s.t. A and B are disjoint).

any two points of A can be connected via a continuous path fully inside A ; and similarly for B).

(This may seem obvious, but it is the famous Jordan curve theorem).

One of these two sets will be bounded, and the other unbounded.

We call the bounded one the area surrounded by γ^* .



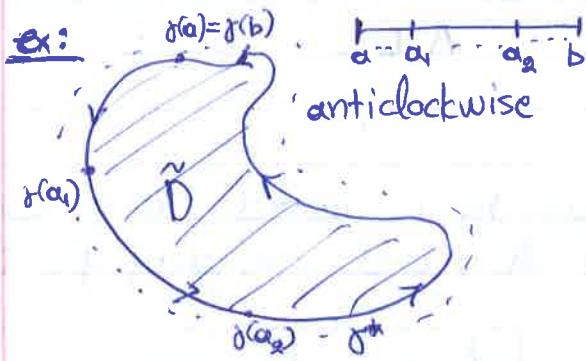
→ Def: A simple, closed curve $\gamma: [a, b] \rightarrow \mathbb{C}$ is called

anticlockwise, if, when we travel on the curve

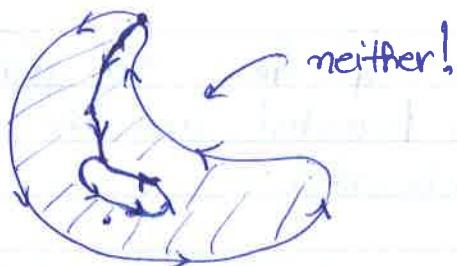
(i.e., when we draw the values $\gamma(x)$ for x running from a to b), then the area γ^* surrounds is always on our left.

(7)

We say that γ is clockwise if, when we travel on the curve (as x runs from a to b), the area that γ^* surrounds is always on our right.

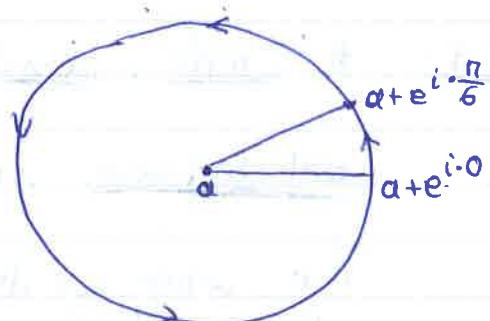


(going once around outer curve anticlockwise, then down to the inner curve, then around the inner curve once clockwise, then back up to the starting point).



(going once around outer curve anticlockwise, then down to the inner curve, then around the inner curve anticlockwise, then back to the starting point).

The circle $C(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$ is an anti-clockwise curve when parametrised as $\gamma(t) = a + r \cdot e^{it}$, $t \in [0, 2\pi]$.



Notice that indeed $C(a, r) = a + r \cdot C(0, 1) = \{a + r \cdot z : z \in C(0, 1)\} = \{a + r \cdot e^{it} : t \in [0, 2\pi]\}$.

(8)

→ Def:Let $\gamma: [\alpha, b] \rightarrow \mathbb{R}$ be a differentiable curveLet $f: \gamma^* \rightarrow \mathbb{C}$. We define
 γ^* is a subset of \mathbb{C}

$$\boxed{\int_{\gamma} f(z) dz} := \int_a^b f(\gamma(t)) \underbrace{d(\gamma(t))}_{\gamma'(t) dt} =$$

see how much it matters

what γ is, rather than just γ^* : If we flipthe orientation (i.e. run from $\gamma(b)$ to $\gamma(a)$), the sign changes! Other than that, the exact parametrisation doesn't matter.

$$= \int_a^b \underbrace{f(\gamma(t)) \cdot \gamma'(t)}_{\in \mathbb{C}} dt .$$

→ Examples: Let $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$:

γ is the unit circle.
 γ is simple,
closed and
anti-clockwise.

- Let $f(z) = z$, $\forall z \in \mathbb{C}$. Then:

$$\int_{\gamma} f(z) dz = \int_0^{2\pi} f(e^{it}) \underbrace{d(e^{it})}_{(e^{it})' dt} = \int_0^{2\pi} e^{it} \cdot i e^{it} dt = i \cdot \int_0^{2\pi} e^{2it} dt = 0.$$

- Let $g(z) = \frac{1}{z}$, $\forall z \neq 0$. Then:

$$\int_{\gamma} g(z) dz = \int_0^{2\pi} g(e^{it}) \cdot \underbrace{d(e^{it})}_{(e^{it})' dt} = \int_0^{2\pi} \frac{1}{e^{it}} \cdot i \cdot e^{it} dt = 2\pi i.$$

(3)

- $h(z) = z^k$, for $k \in \mathbb{Z} \setminus \{-1\}$ (i.e., $\frac{1}{z^2}$, or $\frac{1}{z^3}$, or $\frac{1}{z^4}$, ...)
or z , or z^2 , or z^3 , or ...)

$$\begin{aligned}
 \int_{\gamma} h(z) dz &= \int_0^{2\pi} h(e^{it}) \underbrace{d(e^{it})}_{(e^{it})' dt} = \int_0^{2\pi} e^{ikt} \cdot i \cdot e^{it} dt = \\
 &= i \cdot \int_0^{2\pi} e^{i(k+1)t} dt = i \cdot \int_0^{2\pi} \frac{1}{i(k+1)} \cdot (e^{i(k+1)t})' dt = \\
 &= \frac{1}{k+1} \cdot [e^{i(k+1)t}]_0^{2\pi} = 0.
 \end{aligned}$$

What is the point of all this?

(10)

Goal: understanding $\int \oint f(z) dz$ on closed paths γ ,
for any complex valued f .

Plan: Cauchy's integral theorem $\left\{ \begin{array}{l} \int \oint f(z) dz = 0 \text{ on any closed path } \gamma, \\ \text{when } f \text{ is holomorphic in the area } \gamma \text{ surrounds. (+ conditions).} \end{array} \right.$

Cauchy's integral formula $\left\{ \begin{array}{l} f(z) = \frac{1}{2\pi i} \int \frac{f(w)}{w-z} dw \text{ on any closed path } \gamma \\ \text{that surrounds } z, \text{ s.t. } f \text{ is holomorphic in the area } \gamma \text{ surrounds (+ conditions).} \end{array} \right.$

a way
to express
 $f(z)$

use this to
find expression for $f'(z)$

use this to show
that f' holomorphic

f holomorphic $\rightsquigarrow f$ smooth.

use this to
show that f
equals its Taylor
series on any disc
where f is holomorphic

use this to
show that, if f
is holomorphic on an
annulus (not a disc),
then it can be written
as a Laurent series
(power series
with negative powers too)

$$\int \oint f(z) dz = \int \text{of the Laurent series}$$

= something that only
depends on the points
surrounded by γ where
 f is not holomorphic.

Cauchy's integral theorem

Let $f: D \rightarrow \mathbb{C}$.

↙
a subset
of \mathbb{C}

Then, $\int_{\gamma} f(z) dz = 0,$

for any curve $\gamma: [a, b] \rightarrow D$,
such that :

γ is closed, anticlockwise
(or clockwise),

differentiable,

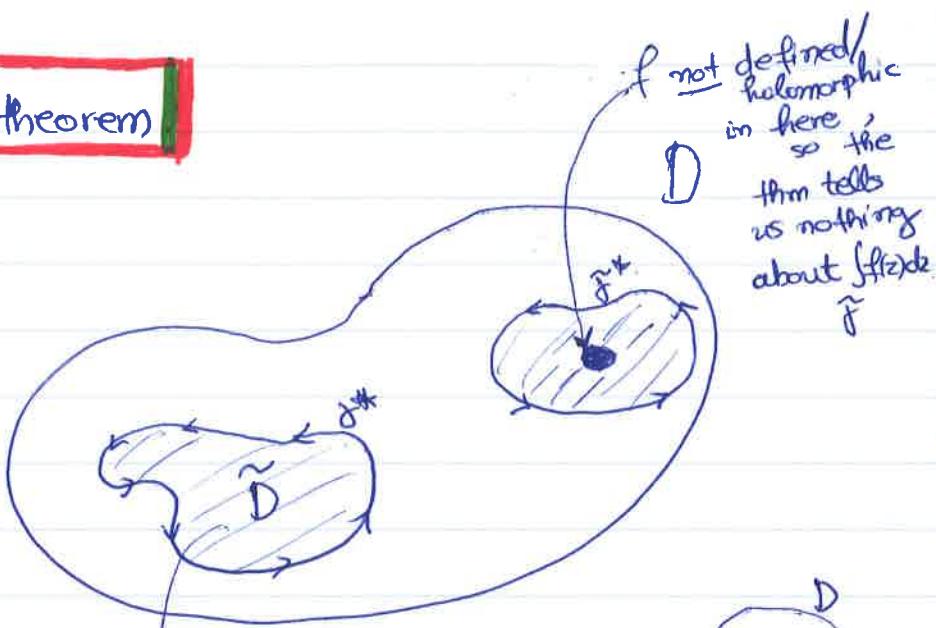
↳ except perhaps
at finitely many pts

simple,

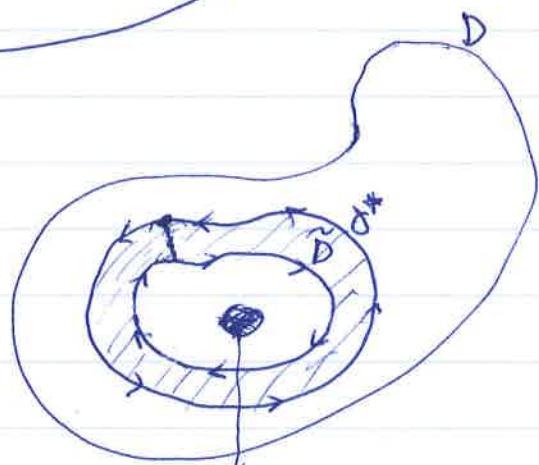
↳ or crosses itself
finitely many times

as long as

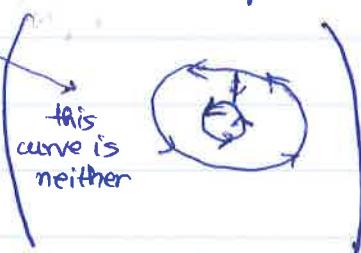
f is holomorphic on γ^* and the area \tilde{D}
that γ surrounds.



if f holomorphic
on γ^* and \tilde{D}
 $\int_{\gamma} f(z) dz = 0$



even if f not defined/holomorphic
here, still $\int_{\gamma} f(z) dz = 0$ if
 f holomorphic on γ^* and \tilde{D} .



①

Proof when $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous on D ,
 γ smooth, D open :

Let γ be as above, anticlockwise. Then:

$$\int_C f(z) dz = \int_{(x+iy) \in \gamma} f(x+iy) d(x+iy) =$$

$\underbrace{(x+iy)}_{\in \mathbb{R}} \underbrace{\epsilon \gamma}_{\in \mathbb{R}} \parallel u(x,y) + i v(x,y) \underbrace{dx + idy}_{dx + idy}$

$$= \int_{(x,y) \in \gamma} u(x,y)(dx + idy) + \int_{(x,y) \in \gamma} i v(x,y)(dx + idy) =$$

$$= \int_{(x,y) \in \gamma} (u(x,y) dx - v(x,y) dy) + i \cdot \int_{(x,y) \in \gamma} (v(x,y) dx + u(x,y) dy)$$

by Green's thm: $\int_C (Pdx + Qdy) \stackrel{\substack{\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x} \\ \text{continuous,} \\ \text{smooth}}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$
 curve C
 anti-clockwise
 area surrounded by C

$$= \iint_D \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \cdot \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy = 0.$$

$\underbrace{\iint_D \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy}_{\substack{\parallel \\ (\text{Cauchy-Riemann conditions})}}$ $\underbrace{\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy}_{\substack{\parallel \\ (\text{Cauchy-Riemann again})}}$

(2)



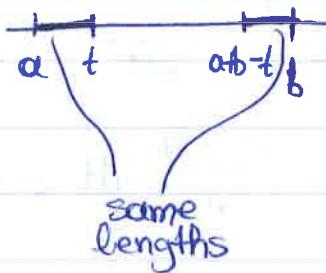
If we reverse the parametrisation $\gamma: [a, b] \rightarrow \mathbb{C}$,

to get the parametrisation $\tilde{\gamma}: [a, b] \rightarrow \mathbb{C}$

$$\tilde{\gamma}(t) = \gamma(a + b - t),$$

then we just flip the orientation of our curve, but keep the same γ^* .

And we see that



$$\int_{\tilde{\gamma}} f(z) dz = - \int_{\gamma} f(z) dz. \text{ (check it!)}$$

So, in the proof of the Cauchy integral theorem for γ clockwise, we have $\int_{\gamma} f(z) dz = - \int_{\tilde{\gamma}} f(z) dz = -0 = 0$.



anti-clockwise



Remember:

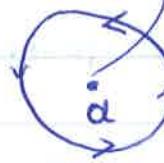
$$\int_{C(0,1)} \left(\frac{1}{z}\right) dz \underset{\substack{\text{holomorphic in } \mathbb{C} \setminus \{0\} \\ \text{the unit circle parametrised by } e^{it} : t \in [0, 2\pi]}}{=} \int_0^{2\pi} \frac{1}{e^{it}} \underbrace{d(e^{it})}_{i e^{it} dt} = 2\pi i \neq 0.$$

(anti-clockwise)

The Cauchy integral theorem doesn't work because $D = D(0, 1)$ contains 0, where $\frac{1}{z}$ is not holomorphic (or even defined).

(3)

$\frac{1}{z-a}$ not defined at a , so Cauchy's integral thm won't work.

 $C(a,1)$:

the unit circle

centered at a ,

parametrised by

 $\delta(t) = a + e^{it}, t \in [0, 2\pi]$

(anti clockwise)

More generally,

$$\begin{aligned}
 \int_{C(a,1)} \frac{1}{z-a} dz &= \int_0^{2\pi} \frac{1}{(a+e^{it})-a} \underbrace{d(e^{it})}_{(e^{it})' dt} = \\
 &= \int_0^{2\pi} \frac{1}{e^{it}} \cdot i \cdot e^{it} dt \\
 &= i \cdot \int_0^{2\pi} 1 dt = 2\pi i \quad (\neq 0).
 \end{aligned}$$

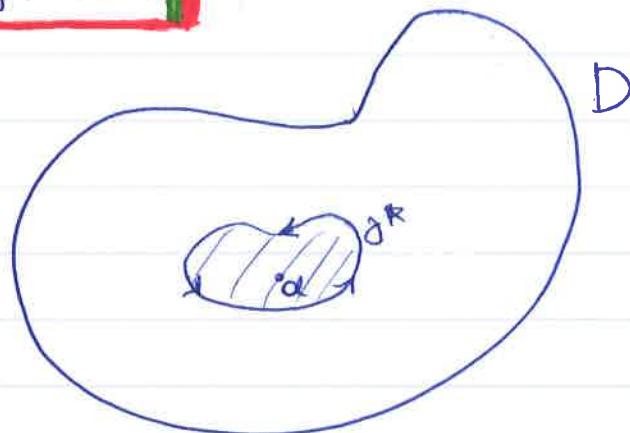
 $C(a,1)$
anticlockwise

We now see a generalisation of this:

→ Cauchy's integral formula:

Let $f: D \rightarrow \mathbb{C}$.

↓
a subset
of \mathbb{C}



Then, for all $a \in D$:

$$f(a) = \frac{1}{2\pi i} \cdot \int_{\gamma} \frac{f(z)}{z-a} dz,$$

for any $f: [\alpha, b] \rightarrow D$ curve, such that:

γ is closed, anticlockwise, differentiable, simple
 (NOT clockwise)
 ↓
 flips the sign
 ↓
 except perhaps
 at finitely many
 points
 ↓
 or crosses itself
 finitely many
 times

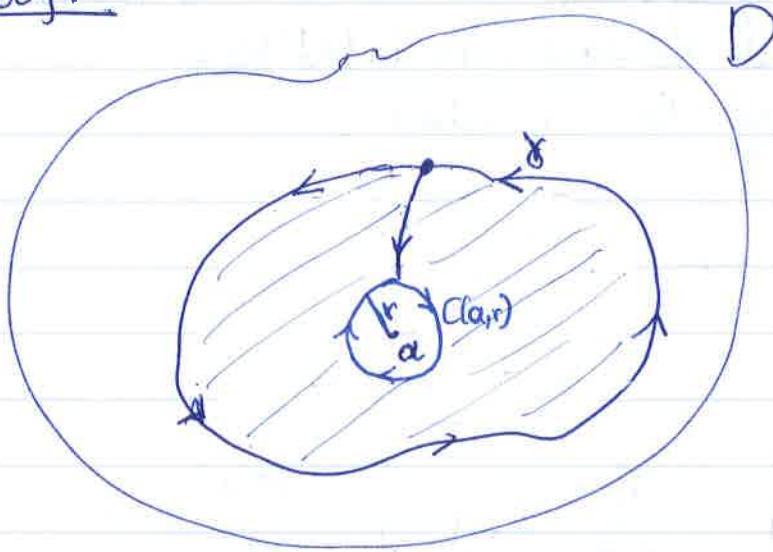
as long as

- a is surrounded by γ
- and f is holomorphic on γ^* and the area that γ surrounds.



We can of course not apply Cauchy's integral theorem to $\frac{f(z)}{z-a}$, as this function is not holomorphic at a (it's not even defined there).

Proof:



Let $a \in D$ and γ a curve as above, surrounding a .

Let $C(a,r)$ be a circle centered at a with radius r , surrounded by γ (the proof will work for all such circles!) We now consider $\tilde{\gamma}$ to be the anticlockwise curve that goes around γ once anticlockwise, goes down to $C(a,r)$ and goes around it once clockwise, and then goes back up to where it started from.

The shaded area is the area surrounded by $\tilde{\gamma}$.

Notice that, since $\frac{1}{z-a}$ is holomorphic on $D \setminus \{a\}$,

and $f(z)$ is holomorphic on D , then

$\frac{f(z)}{z-a}$ is holomorphic on $D \setminus \{a\}$. Since a is not

(6)

contained in γ^* and the area surrounded by γ , $\frac{f(z)}{z-a}$ is holomorphic on γ^* and the area surrounded by γ .
 So, Cauchy's integral theorem gives :

$$\int_{\gamma} \frac{f(z)}{z-a} dz = 0. \quad \text{But:}$$

$$\int_{\gamma} \frac{f(z)}{z-a} dz = \int_{\gamma \text{ anticlockwise}} \frac{f(z)}{z-a} dz + \int_{A \rightarrow B} \frac{f(z)}{z-a} dz +$$

$$+ \int_{C(a,r)} \frac{f(z)}{z-a} dz + \int_{B \rightarrow A} \frac{f(z)}{z-a} dz \rightarrow$$

$$\rightarrow \int_{\gamma} \frac{f(z)}{z-a} dz = \int_{C(a,r)} \frac{f(z)}{z-a} dz = \int_0^{2\pi} \frac{f(a+re^{it})}{a+re^{it}-a} i e^{it} dt$$

$$= i \cdot \int_0^{2\pi} f(a+re^{it}) dt, \quad \text{for } r \text{ small!}$$

$$\begin{aligned} \text{So, } \int_{\gamma} \frac{f(z)}{z-a} dz &= \lim_{r \rightarrow 0} i \cdot \int_0^{2\pi} f(a+re^{it}) dt \\ &= i \cdot \int_0^{2\pi} \lim_{r \rightarrow 0} f(a+r \cdot e^{it}) dt \end{aligned}$$

because $C(a,r)$ is surrounded by γ for all these r 's.

f continuous

(7)

$$= i \cdot \int_0^{2\pi} f(a) dt = 2\pi i \cdot f(a).$$

⚠ Think of Cauchy's integral formula as a way to rewrite f !

→ **Corollary:** Let $f: U \rightarrow \mathbb{C}$ be holomorphic on U .
 \uparrow
 an open
 $\subseteq \mathbb{C}$

Then, for all $z \in U$

$$f'(z) = \frac{1}{2\pi i} \cdot \int_U \frac{f(w)}{(w-z)^2} dw,$$

for any curve $\gamma: [a, b] \rightarrow \underline{U}$
closed, anticlockwise, differentiable, simple,

with

z surrounded by γ
 and f holomorphic in the area γ surrounds

(if you are interested).

Sketch of proof: Let $z_0 \in U$, and γ a curve as
 above in U , surrounding z_0 . for z close to z_0 in U ,
 γ also surrounds z . So:

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\frac{1}{2\pi i} \cdot \int_{\gamma} \frac{f(w)}{w-z} dw}{z - z_0} - \frac{\frac{1}{2\pi i} \cdot \int_{\gamma} \frac{f(w)}{w-z_0} dw}{z - z_0} =$$

8

$$\begin{aligned}
 &= \frac{1}{2\pi i} \cdot \int_{\gamma} \frac{\frac{1}{w-z} - \frac{1}{w-z_0}}{z-z_0} dw = \frac{1}{2\pi i} \cdot \int_{\gamma} f(w) \cdot \frac{\frac{1}{w-z} - \frac{1}{w-z_0}}{(z-w)-(z_0-w)} dw \\
 &= \frac{-1}{2\pi i} \cdot \int_{\gamma} f(w) \cdot \frac{\frac{1}{z-w} - \frac{1}{z_0-w}}{(z-w)-(z_0-w)} dw
 \end{aligned}$$

So, we are interested in the limit

$$\lim_{z \rightarrow z_0} \frac{-1}{2\pi i} \cdot \int_{\gamma} f(w) \cdot \frac{\frac{1}{z-w} - \frac{1}{z_0-w}}{(z-w)-(z_0-w)} dw \quad \text{conditions to put } \lim \text{ inside } \int$$

$$\begin{aligned}
 &= -\frac{1}{2\pi i} \cdot \int_{\gamma} f(w) \cdot \lim_{\substack{z \rightarrow z_0 \\ z-w \rightarrow z_0-w}} \frac{\frac{1}{(z-w-\tilde{w})} - \frac{1}{z_0-w}}{(z-w)-(z_0-w)} dw = \\
 &\quad = \left(\frac{1}{w}\right)' \Big|_{\tilde{w}=z_0-w} = \frac{-1}{(z_0-w)^2}
 \end{aligned}$$

$$= \frac{1}{2\pi i} \cdot \int_{\gamma} \frac{f(w)}{(w-z_0)^2} dw$$

→ Corollary:

If $f: U \xrightarrow{\text{open } \subseteq \mathbb{C}}$ is holomorphic on U ,
 then f is infinitely differentiable on U .

(if you are interested)

Proof: f holomorphic $\rightarrow f'(z) = \frac{1}{2\pi i} \cdot \int_{\gamma} \frac{f(w)}{(w-z)^2} dw$, for

any curve γ as above. Just like in the proof above,

(9)

we want to understand

$$\frac{f'(z) - f'(z_0)}{z - z_0} \quad \text{as } z \rightarrow z_0. \quad \text{We imitate the}$$

above proof, and we see that the limit

exists, and equals $\frac{2!}{2\pi i} \cdot \int \frac{f(w)}{(w-z_0)^3} dw$.

$$\therefore f''(z_0) = \frac{2!}{2\pi i} \cdot \int \frac{f(w)}{(w-z_0)^3} dw.$$

We now want to understand

$$\frac{f''(z) - f''(z_0)}{z - z_0} \quad \text{as } z \rightarrow z_0. \quad \text{We work as above, and so on.}$$

In the end, we show that f is infinitely differentiable at z_0 , and

in complex numbers

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \cdot \int \frac{f(w)}{(w-z_0)^{n+1}} dw, \quad \text{for all } z_0 \in U, \text{ for}$$

any curve $\gamma: [a, b] \rightarrow U$ closed, clockwise, differentiable, simple, surrounding z_0 , with f holomorphic in the area γ surrounds.

10

→ Corollary: Let $f: D(z_0, r) \rightarrow \mathbb{C}$ holomorphic.

for some

$$\text{Then, } f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(z_0)}{k!} (z-z_0)^k, \quad \forall z \in D(z_0, r)$$

i.e., f equals its

Taylor series in the largest disc around z_0 where it is holomorphic!

if you are interested
Proof:



Let $z \in D(z_0, r)$.

Let $C(z_0, r')$ be a circle centered at z_0 and containing z . We parametrise the circle $C(z_0, r')$ so that it is anti clockwise.

Notice that f is holomorphic on $C(z_0, r')$ and the area that $C(z_0, r')$ surrounds. So, by Cauchy's integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \cdot \int \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \int \frac{f(w)}{(w-z_0)-(z-z_0)} dw = \\ &= \frac{1}{2\pi i} \cdot \int_{C(z_0, r')} \frac{\frac{f(w)}{f(w)}}{(w-z_0) \cdot \left(1 - \frac{z-z_0}{w-z_0}\right)} dw = \end{aligned}$$

(11)

$$= \frac{1}{2\pi i} \cdot \int_{C(z_0, r')} \frac{f(w)}{w-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}} dw =$$

$$\left| \frac{z-z_0}{w-z_0} \right| < 1, \text{ as } |z-z_0| < |w-z_0|,$$

since $z \in D(z_0, r')$

while $w \in C(z_0, r')$. Expand in geometric series!

$$= \frac{1}{2\pi i} \int_{C(z_0, r')} \frac{f(w)}{w-z_0} \cdot \sum_{n=0}^{+\infty} \left(\frac{z-z_0}{w-z_0} \right)^n dw \quad \begin{matrix} \Delta \text{ conditions} \\ \text{to take } S \\ \text{out of } \lim \end{matrix}$$

$$= \sum_{n=0}^{+\infty} \left(\frac{1}{2\pi i} \int_{C(z_0, r')} \frac{f(w)}{(w-z_0)^{n+1}} dw \right) \cdot (z-z_0)^n =$$

$\underbrace{\frac{f^{(n)}(z_0)}{n!}}$, by the proof of
the previous Corollary.

$$= \sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z-z_0)^n.$$

(12)

On the other hand, the following also holds
(we will not see a proof of this):

→ If a series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges inside a disc $D(z_0, r)$, then $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ is holomorphic in $D(z_0, r)$.

 $r > 0$ 

The last two statements imply that

the holomorphic functions on a disc are exactly
the power series centered at the center of the disc,
with domain the disc, that converge on the disc.

We have by now seen that, if f is holomorphic on \mathbb{C} ,
then it will be equal to its Taylor series around
any center, on the whole of \mathbb{C} .

Suppose $f, g : \mathbb{C} \rightarrow \mathbb{C}$ holomorphic. Suppose that

$f = g$ on a disc $D(a, r)$, for some $a \in \mathbb{C}$ and $r > 0$.

Then, the above implies that $f = g$ everywhere!

Indeed, $f = g$ on $D(a, r) \Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k = \sum_{k=0}^{\infty} \frac{g^{(k)}(a)}{k!} (z-a)^k$

for $z \in D(a, r)$, so

(13)

the power series have the same coefficients.

However,

$$\sum_{k=0}^{+\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k = f(z) \quad \forall z \in \mathbb{C},$$

||

as f holomorphic on the whole of \mathbb{C} ,

and similarly

$$\sum_{k=0}^{+\infty} \frac{g^{(k)}(a)}{k!} (z-a)^k = g(z) \quad \forall z \in \mathbb{C}.$$

So, $f(z) = g(z) \quad \forall z \in \mathbb{C}$!

This is a major theorem in complex analysis, that generalises to situations where f, g are not holomorphic on the whole of \mathbb{C} :

→
↓
if you
are
interested.

Identity theorem :

Let $U \subseteq \mathbb{C}$ be connected and open.

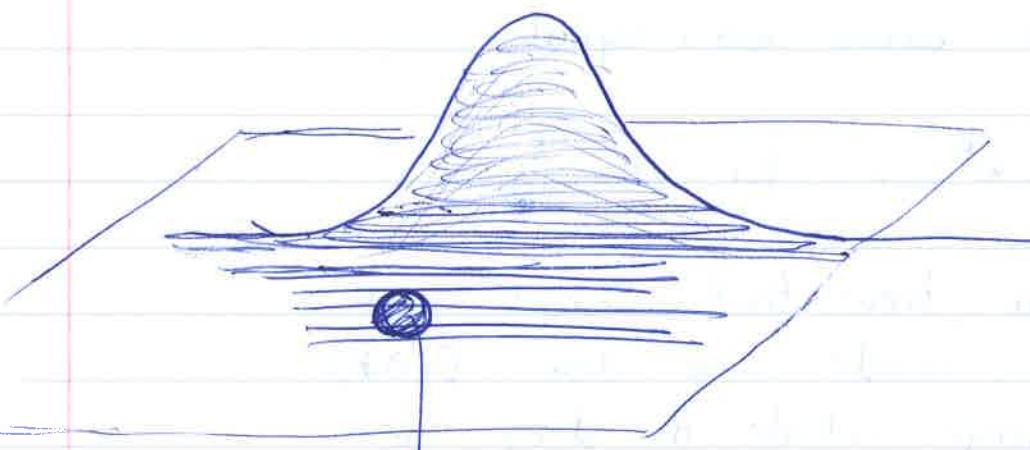
(e.g. \mathbb{C} , $\mathbb{C} \setminus \{0\}$, $\mathbb{D} \setminus \{\text{half-line}\}$,
 $\mathbb{D} \setminus \{\text{closed disc}\}$)

Let $f, g : U \rightarrow \mathbb{C}$ holomorphic.

If $f=g$ on some disc inside U ,
then $f=g$ on the whole of U .



Notice that this means that bump holomorphic functions don't exist! (while of course smooth bump functions: $\mathbb{R} \rightarrow \mathbb{R}$ exist).



if this function was holomorphic,
then, since it is 0 on a disc, it would
have to be 0 everywhere!

Now, what if a function is, say, holomorphic everywhere, except at a point $a \in \mathbb{C}$? It is certainly holomorphic everywhere in the disc $D = D(0, |a|)$, so f equals a power series in D . For instance, the function $f(z) = \frac{1}{1-z}$ is holomorphic on $\mathbb{C} \setminus \{1\}$, and $f(z) = 1 + z + z^2 + \dots$ in $D(0, 1)$.



However, what happens outside D ? Here is where Laurent series come in.

→ Consider a series of the form

$$\sum_{k=0}^{+\infty} a_k \cdot \frac{1}{(z-a)^k}, \text{ where } \{a\}, \text{ for } a \text{ some fixed complex number.}$$

Then, the above series equals

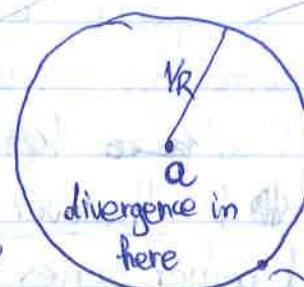
$$\sum_{k=0}^{+\infty} a_k w^k, \text{ for } w = \frac{1}{z-a}.$$

Since we know that $\sum_{k=0}^{+\infty} a_k w^k$ will converge inside some disc $D(0, R)$ and diverge outside the disc, we

have that $\sum_{k=0}^{+\infty} a_k w^k$ converges $\Leftrightarrow |w| < R$,

$$\text{so } \sum_{k=0}^{+\infty} a_k \frac{1}{(z-a)^k} \text{ converges} \Leftrightarrow |z-a| > \frac{1}{R}$$

i.e. $\Leftrightarrow z$ is outside the disc $D(a, \frac{1}{R})$!



$$\sum_{k=0}^{+\infty} a_k \frac{1}{(z-a)^k}$$

converges for
 z here.

anything can happen
for z on this circle.

So, such series have the reverse behaviour than power series.

→ Now, consider a series of the form

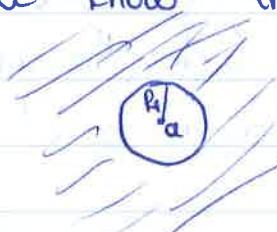
$$\sum_{k=1}^{+\infty} a_k \cdot \frac{1}{(z-a)^k} + \sum_{k=0}^{+\infty} b_k (z-a)^k : \text{a Laurent series}$$

$$= b_0 + b_1(z-a) + b_2(z-a)^2 + \dots + \frac{a_1}{z-a} + \frac{a_2}{(z-a)^2} + \frac{a_3}{(z-a)^3} + \dots,$$

which we sometimes denote by

$$\dots + \frac{a_3}{(z-a)^3} + \frac{a_2}{(z-a)^2} + \frac{a_1}{z-a} + b_0 + b_1(z-a) + b_2(z-a)^2 + \dots)$$

We know that $\sum_{k=1}^{+\infty} a_k \cdot \frac{1}{(z-a)^k}$ converges when $|z-a| > R_1$,



for some $R_1 > 0$, i.e.
outside the disc $D(a, R_1)$.

And $\sum_{k=0}^{+\infty} b_k \cdot (z-a)^k$ converges when $|z-a| < R_2$ for some $R_2 \geq 0$, i.e.

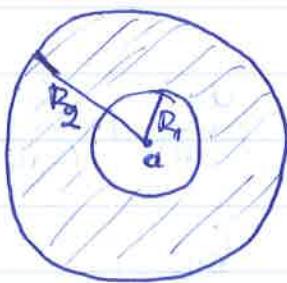


inside the disc $D(a, R_2)$.

(17)

So, the whole Laurent series converges in the intersection of these two areas,

which is the annulus $A(R_1, R_2)$ (notice that,



if $R_1 > R_2$, this annulus is empty and the Laurent series doesn't converge for any $z \in \mathbb{C}$).

We have thus shown that any Laurent series converges inside an annulus,

with inner radius the radius of the smallest disc outside which the series of negative powers converges,

and outer radius the radius of the largest disc inside which the series of positive powers converges.

Theorem (Laurent):

Let

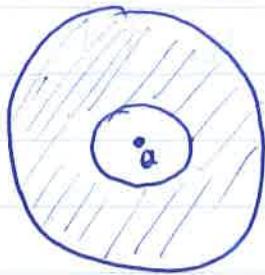
when $R_1=0$, this annulus is just the punctured disc $D(a, R_2) \setminus \{a\}$!

①

$0 \leq R_1 < R_2$. Fix $a \in \mathbb{C}$.

Consider the annulus

$$A(R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z-a| < R_2\}.$$

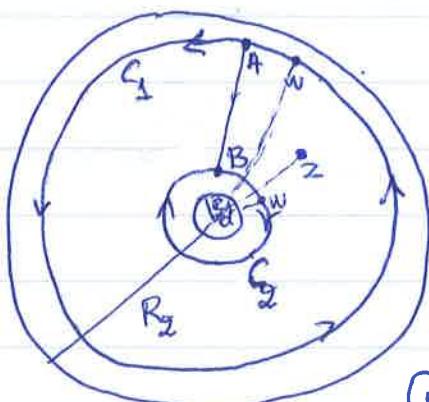


Then, any f that is holomorphic on $A(R_1, R_2)$ can be written as a Laurent series on $A(R_1, R_2)$, centered at a . I.e. :

$$f(z) = \dots + \frac{a_{-3}}{(z-a)^3} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

for some coefficients $\dots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots$ in \mathbb{C} .

Proof: The proof is very similar to that for writing f as a power series when it is holomorphic on a disc. Let $z \in A(R_1, R_2)$.



Let C_1, C_2 be two circles centered at a , with z between them. \nearrow inside $A(R_1, R_2)$

Since f is holomorphic on the two circles and in the area between them, Cauchy's integral formula gives:

(2)

$$f(z) = \frac{1}{2\pi i} \int_A \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \int_B \frac{f(w)}{w-z} dw$$

~~A > B~~

anti-clockwise

$$+ \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \int_{B \cap A} \frac{f(w)}{w-z} dw$$

~~B < A~~

clockwise

$$= \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z} dw$$

anti-clockwise clockwise

Now, we use the same idea as in p. 11, Lecture 10:

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-a)-(z-a)} = \frac{f(w)}{(w-a) \cdot \left(1 - \frac{z-a}{w-a}\right)}$$

Notice that, when $w \in C_1$, then $|z-a| < |w-a|$

$\rightarrow \frac{|z-a|}{|w-a|} < 1$, so we can

$$\text{write } \frac{1}{1 - \frac{z-a}{w-a}} = 1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \dots$$

and finish up as in p. 11, Lecture 10.

However, when $w \in C_2$, then $|z-a| > |w-a|$, so we can't

(3)

work as above. In that case, we see

$$\text{that: } \frac{f(w)}{w-a} = -\frac{f(w)}{z-w} = -\frac{f(w)}{(z-a)-(w-a)} =$$

$$= -\frac{f(w)}{(z-a)\left(1-\frac{w-a}{z-a}\right)} = -\frac{f(w)}{z-a} \cdot \left(1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \dots\right)$$

↙
see how negative
powers of $z-a$ appear!

And then we finish up as in p.11, Lecture 10. ■



See how the above proof gives all the coefficients, too (as in p.11, Lecture 10); they are

$$(*) \quad a_k = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{k+1}} dw, \quad k \in \mathbb{Z};$$

at a in the annulus doesn't matter

for $k \geq 1$, these are not $\frac{f^{(k)}(a)}{k!}$ any more, as f may not be differentiable at a

this is just a generalisation of what happens when f is holomorphic!

Remember now, our goal was to understand $\int f(z)dz$ for general $f: \mathbb{C} \rightarrow \mathbb{C}$, for any γ closed curve (differentiable, simple).

Notice that such an integral of f appears above
 for $k = -1$: $a_{-1} = \frac{1}{2\pi i} \cdot \int_{C_1} f(w) dw$!

④

by \star

In other words, if f is holomorphic on

an annulus centered at $a \in \mathbb{C}$, and C_a
 is a circle centered at a , inside the
annulus, then

$$\int_{C_a} f(z) dz = 2\pi i \cdot a_{-1}$$

this makes us suspect that

$$\int f(z) dz \text{ for } \gamma \text{ closed}$$

may actually only depend
 on such coefficients a_{-1}
 with respect to all
 singularities of f ,
 surrounded by γ !

the coefficient of $\frac{1}{z-a}$ in

the Laurent series expansion of
 f in the annulus

⚠ if we change the annulus, to
 include more singularities, a_{-1} will change.
 (see discussion on p. 679, 680
 of the textbook.)

Now that we know, however, that holomorphic
 functions on annuli can be written as
 Laurent series, we can get a much better result:

(5)

→ Def: Let $f: D \xrightarrow{\text{ct}} \mathbb{C}$. Suppose that a is an

isolated singularity of f
(i.e. f is holomorphic in some punctured disc $D(a, r) \setminus \{a\}$)



We know that

$$f(z) = \dots + \frac{a_{-3}}{(z-a)^3} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$$

an annulus

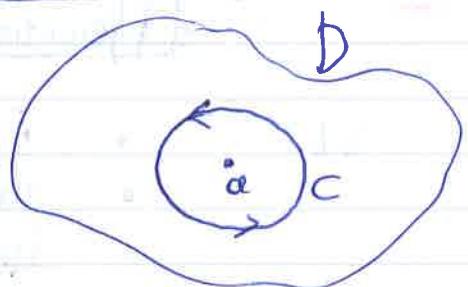
on any punctured disc $D(a, r) \setminus \{a\}$ that

contains no other singularity of f . We define

the residue $\text{Res}(f; a)$ of f at a to be a_{-1} .

→ Observation: Let f and a be as above.

Let C be an anticlockwise circle in D , centered at a , that surrounds no other singularity of f .



$$\int_C (z-a)^k dz = 0 \quad \forall k > 0$$

Cauchy's integral theorem

$$\int_C \frac{1}{(z-a)^k} dz = 0 \quad \forall k \neq 1, \quad \text{and}$$

$$\int_C \frac{1}{z-a} dz = 2\pi i. \quad \text{So:}$$

$$\int_C f(z) dz = a_{-1} \cdot \int_C \frac{1}{z-a} dz = 2\pi i \cdot a_{-1} = 2\pi i \cdot \text{Res}(f, a)$$

(This also follows from p. 4, but it is much more straightforward this way).

(6)

To sum up :

$$\int_C f(z) dz = 2\pi i \cdot \text{Res}(f; a)$$

for any anticlockwise circle centered at a
that contains no other singularities of f

(Notice that the above holds even if a itself is not a singularity of f ; then the residue will be 0, so we get Cauchy's integral formula for this C).

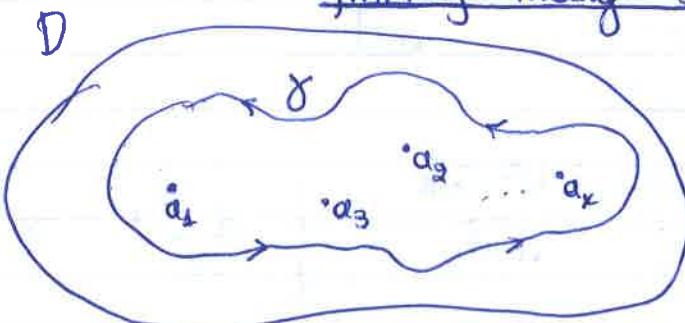
→ The residue theorem:

Let $f: D \xrightarrow{\subset \mathbb{C}} \mathbb{C}$.

Then, for any curve $\gamma: [a, b] \rightarrow D$

differentiable, simple, closed, anticlockwise,

- s.t.
- f holomorphic on γ^*
 - the area γ surrounds contains only finitely many singularities of f ,



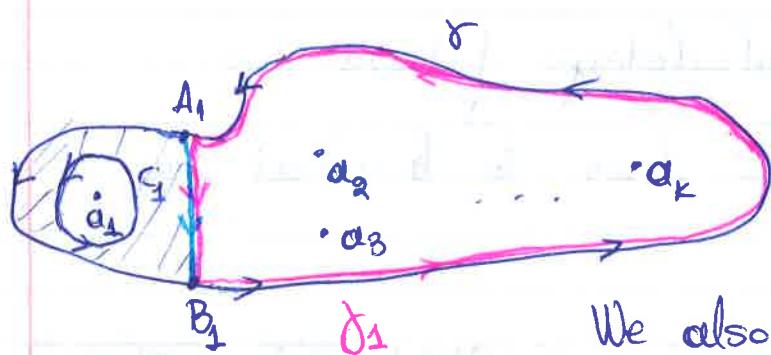
then

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{a_i \text{ the singularities of } f \text{ that surrounds}} \text{Res}(f; a_i)$$

a_i the singularities of f that surrounds.

(7)

Sketch of proof: Since γ surrounds only finitely many singularities $\alpha_1, \alpha_2, \dots, \alpha_k$ of f , these singularities are all isolated. We now notice that we can use the previous Observation to "cut off" these singularities one by one. In particular, let's start with α_1 :



We draw a curve (the light blue one) with endpoints A_1, B_1

that separates α_1 from the rest of the singularities.

We also pick a circle G centered at α_1 as in the picture; by Cauchy's

integral theorem (as f has no singularities in the shadowed area), we obtain:

$$\int_{\gamma} f(z) dz + \int_{B_1 \rightarrow A_1 \text{ on light blue curve}} f(z) dz = \int_G f(z) dz \stackrel{\text{Observation}}{=} 2\pi i \cdot \text{Res}(f; \alpha_1).$$

we want to replace this

A₁ → B₁ on γ

So:

$$\int_{\gamma} f(z) dz = \int_{B_1 \rightarrow A_1 \text{ on } \gamma} f(z) dz + \int_{A_1 \rightarrow B_1 \text{ on } \gamma} f(z) dz \stackrel{\text{by } \textcircled{*}}{=} \int_{B_1 \rightarrow A_1 \text{ on } \gamma} f(z) dz + 2\pi i \cdot \text{Res}(f; \alpha_1) + \int_{A_1 \rightarrow B_1 \text{ on light blue curve}} f(z) dz =$$

(8)

$$= 2\pi i \operatorname{Res}(f; a_1) + \int_{\gamma} f(z) dz$$

γ_1
↓

γ_1 surrounds
one singularity
fewer. We "cut off"
 a_2 from inside γ_1 ,
etc ... so we eventually get

the residue theorem!

→ So, the problem of calculating $\int f(z) dz$ on a closed curve γ boils down to calculating residues!

→ Techniques to find $\operatorname{Res}(f; a)$:

an isolated singularity of f .

(I) By definition of $\operatorname{Res}(f; a)$, we know that

$$\operatorname{Res}(f; a) = a_1, \text{ the coefficient of } \frac{1}{z-a}$$

in the Laurent series of f around a , in a punctured disc $D(a, r) \setminus \{a\}$ that contains no other singularities of f . So, all we need is to find

that Laurent series, and pick out a_1 .

(g)

ex:

$$f(z) = \frac{(z-2)(z+5)}{z-1}$$

Notice that f has only 1 as a singularity, so we know

we can actually write f as a Laurent series

$$\dots + \frac{a_{-3}}{(z-1)^3} + \frac{a_{-2}}{(z-1)^2} + \frac{a_{-1}}{z-1} + a_0 + a_1(z-1) + a_2(z-2)^2 + \dots$$

on the whole of $\mathbb{C} \setminus \{1\}$. To find this Laurent series, we can express first $(z-2)(z+5)$ as a power series centered at 1 (we can do this, as this function is holomorphic on \mathbb{C}),

and then divide each term with $z-1$:

$$(z-2)(z+5) = ((z-1)-1) \cdot ((z-1)+6) =$$

$$= (z-1)^2 + 6(z-1) - (z-1) - 6 = (z-1)^2 + 5(z-1) - 6 \quad \text{if } z \in \mathbb{C},$$

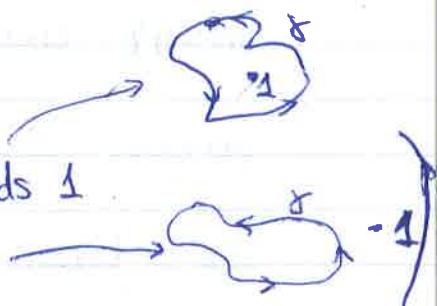
power series centered
at 1

$$\text{so } f(z) = \frac{(z-2)(z+5)}{z-1} = (z-1) + 5 - \frac{6}{z-1}, \quad \text{if } z \in \mathbb{C} \setminus \{1\}.$$

the Laurent series
expression of f
around 1

$$\text{So, } \operatorname{Res}(f; 1) = -6.$$

(and thus $\int \frac{(z-2)(z+5)}{z-1} dz = \begin{cases} -6, & \text{if } \gamma \text{ surrounds 1} \\ 0, & \text{otherwise} \end{cases}$)



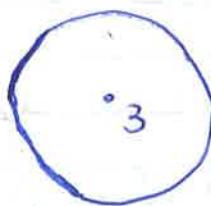
(P)

This is very simple. However, it may not always be so easy to find the Laurent series expression of f around any centre. For instance, let's modify our example a little; consider

$$f(z) = \frac{(z-2)(z+5)}{(z-1)(z-3)^2} \quad \text{This } f \text{ has } \underline{\text{two}}$$

singularities, at 1 and 3. It is of course

true (by definition of residue) that



$\text{Res}(f; 1) = a_{-1}$ in the Laurent series expression of f around 1 on any punctured disc $D(1, r) \setminus \{1\}$ that doesn't contain 3.

(and similarly, $\text{R}(f; 3) = a_{-1}$ in the Laurent series expression of f around 3 on any punctured disc $D(3, r) \setminus \{3\}$ that doesn't contain 1). Let us find $\text{Res}(f; 1)$ with the previous technique:

We know that $\frac{(z-2)(z+5)}{(z-3)^2} \cdot g(z)$ can be written as a

(11)

power series centered at 1, on any disc $D(1, r)$

that doesn't contain 3. To find this series, we can, for instance,

find $\frac{g^{(k)}(1)}{k!} + k \in \mathbb{N}$; then !

$$g(z) = g(1) + g'(1)(z-1) + \frac{g''(1)}{2!}(z-1)^2 + \dots$$

Or, we can write $\frac{(z-2)(z+5)}{(z-3)^2} = \frac{(\underbrace{z-1}_{w}-1)(\underbrace{z-3}_{w}+6)}{(\underbrace{z-1}_{w}-2)^2}$,

and do long division. Both these methods will lead us to $\text{Res}(f; 1)$, but these calculations are a bit complicated. The main observation is that

we don't need the whole Laurent series of f around 1;

we just need a_{-1} . And, since

$$g(z) = g(1) + g'(1)(z-1) + \frac{g''(1)}{2!}(z-1)^2 + \dots,$$

$$\text{we have that } f(z) = \frac{g(z)}{z-1} = \frac{g(1)}{z-1} + g'(1) + \frac{g''(1)}{2!}(z-1) + \dots,$$

$$\text{so } \text{Res}(f; 1) = g(1) =$$

$$= \frac{(1-2)(1+5)}{(1-3)^2} = \frac{-6}{(-2)^2} = -\frac{3}{2} !$$

This generalises to the following faster technique :

21 Sep 2016 ①

Lecture 12:

→ Let f be holomorphic in a disc $D(a,r)$.

We say that a is a root of f of order m

if $f(z) = a_m(z-a)^{m \geq 0} + a_{m+1}(z-a)^{m+1} + \dots$

with $a_m \neq 0$,

i.e. if the power series of f in $D(a,r)$ starts

with power $(z-a)^{m \geq 0}$

ex: 1 is a root of order 1 for $(z-1)$,

1 " " " " " 2 for $(z-1)$,

1 " " " " " 3 for $(z-1)^3 \cdot (z-5)^6$,

5 " " " " " 6 for $(z-1)^3 (z-5)^6$,

0 " " " " " 1 for $\sin z$

In general:

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$= z \cdot \left(1 - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

If $f(z) = (z-a)^{m \geq 0} g(z)$, where $g(a) \neq 0$ and g holomorphic

at (and around) a ,

then a is a root of f of order m .

(2)

Def: Let f be holomorphic in a punctured disc $D(a,r) \setminus \{a\}$ around a .

We say that a is a pole of f of order m another word for singularity

$$\text{if } f(z) = \frac{a_m}{(z-a)^m} + \frac{a_{m+1}}{(z-a)^{m-1}} + \dots + \frac{a_1}{z-a} + a_0 + a_1(z-a) + \dots$$

with $a_m \neq 0$,

i.e. if the Laurent series of f in $D(a,r) \setminus \{a\}$ starts

with power $\frac{1}{(z-a)^m} \geq 0$

ex: 1 is a pole of order

1 for $\frac{1}{z-1}$,

1 " " " " " 2 for $\frac{1}{(z-1)^2}$,

1 " " " " " 3 for $\frac{(z-5)^6}{(z-1)^3}$,

0 " " " " " 0 for $\frac{\sin z}{z}$!

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

converges (to $\frac{\sin z}{z}$)

$\neq 0$, so its disc of convergence
has to include 0, so it is
holomorphic on the whole of \mathbb{C} !

In general:

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$

If $f(z) = \frac{g(z)}{(z-a)^m}$, where $g(a) \neq 0$ and g is holomorphic at and around a , then a is a pole of f of order m .

(3)

If $f(z) = \frac{g(z)}{h(z)}$, with g, h holomorphic at and around α ,

and α is a root of order m for g

and a root of order n for h , and $n \geq m$,

then it is a pole of order $m-n$ for f .

(and of course, if $m > n$, α is a root for f of order $m-n$).

The reason is that then $f(z) = \frac{(z-\alpha)^m \cdot g_1(z)}{(z-\alpha)^n g_2(z)} =$

$$= -\frac{\left(\frac{g_1(z)}{g_2(z)}\right)}{(z-\alpha)} \quad (= (z-\alpha)^{m-n} \cdot \left(\frac{g_1(z)}{g_2(z)}\right)),$$

where g_1, g_2 are holomorphic at and around α ,

and $g_1(\alpha) \neq 0, g_2(\alpha) \neq 0 \Rightarrow \frac{g_1}{g_2}$ is holomorphic at and around α , with $\frac{g_1}{g_2}(\alpha) = \frac{g_1(\alpha)}{g_2(\alpha)} \neq 0$

II Let f be a complex function, and let α be an isolated singularity of f . To find $\text{Res}(f; \alpha)$:

(i) Find the order m of the pole α .

Usually, $f(z) = \frac{g(z)}{h(z)}$, for g, h holomorphic at and around α . We write $g(z) = (z-\alpha)^k g_1(z)$ and $h(z) = (z-\alpha)^{k_1} h_1(z)$, where g_1, h_1 hol. at α and α is not a root of g_1, h_1 . Then, α is a pole of order $k - k_1$ for f .

(4)

$$(ii) \text{Res}(f; \alpha) = \frac{1}{(m-1)!} (z-\alpha)^m \cdot f(z) \Big|_{z=\alpha}^{(m-1)}$$

Proof: Since α is a pole of f of order m , we have that

$$f(z) = \frac{a_m}{(z-\alpha)^m} + \frac{a_{m-1}}{(z-\alpha)^{m-1}} + \dots + \frac{a_1}{z-\alpha} + a_0 + a_1(z-\alpha) + \dots$$

So, $(z-\alpha)^m \cdot f(z) = a_m + a_{m-1}(z-\alpha) + \dots + a_1(z-\alpha)^{m-1} + \dots$

holomorphic
on a disc centered
at α

this is the Taylor
expansion of f around α

$$\text{so } a_{-1} = \frac{(z-\alpha)^m \cdot f(z)}{(m-1)!} \Big|_{(z-\alpha)^m}$$

truly: substitution in power series for $(z-\alpha)^m f(z)$, not $(z-\alpha)^m f(z)$ itself, which may not be defined at α . See later discussion.

ex: Find $\text{Res}(f; 1)$, for $f(z) = \frac{\sin(z-1)}{(z-1)^4 \cdot (z-5)}$:

We see that f has a singularity (pole) at $z=1$. Now:

$$\sin(z-1) = (z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} - \frac{(z-1)^7}{7!} \left(= (z-1) \cdot \left(1 - \frac{(z-1)^2}{3!} + \dots \right) \right),$$

so 1 is a root of $\sin(z-1)$ of order 1.

Also, 1 is a root of $(z-1)^4 (z-5)$ of order 4

So, 1 is a pole of f of order 3.

(5)

Thus, to find $\text{Res}(f; 1)$, we multiply $f(z)$

with $(z-1)^3$, differentiate 2 times, evaluate at 1, and divide by $2!$:

$$\text{Res}(f; 1) = \frac{(z-1)^3 f(z))^{(2)}|_{z=1}}{2!}.$$

It may be the case that we are not sure what order exactly a pole of f has; but maybe we know that it is at most n , for some $n \in \mathbb{N}$. Then, the proof for the technique above works again, to give:

III Let f be a complex function, and let a be an isolated singularity of f . If we know that a has order at most n as a pole, then:

$$\text{Res}(f; a) = \frac{1}{(n-1)!} \cdot \left((z-a)^n \cdot f^{(n)}(a) \right)^{(n-1)}|_{z=a}.$$

Proof: Just like for technique II, as in this case

$$f(z) = \frac{\alpha_n}{(z-a)^n} + \frac{\alpha_{n-1}}{(z-a)^{n-1}} + \dots + \frac{\alpha_1}{z-a} + \alpha_0 + \alpha_1(z-a) + \dots$$

(6)

The only difference is that now maybe $a_m = 0$, i.e. maybe the Laurent series starts a little later; but that makes absolutely no difference in the proof. ■



Notice that techniques \textcircled{II} and \textcircled{III} work only when a is a pole of f of finite order;

in other words, when the Laurent series of f around a starts from some minimal power $(z-a)^{-m}$:

$$f(z) = \frac{a-m}{(z-a)^m} + \frac{a-m+1}{(z-a)^{m-1}} + \dots + \frac{a-1}{z-a} + a_0 + a_1(z-a) + \dots$$

There are cases though where this doesn't happen:

for instance, $e^{1/z}$, $z \in \mathbb{C} \setminus \{0\}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, and

$$e^{1/z} = 1 + \frac{\left(\frac{1}{z}\right)^1}{1!} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \dots =$$

$$= \dots + \underbrace{\frac{1}{n!} \cdot \frac{1}{z^n}}_{\text{infinitely many negative powers of } z} + \frac{1}{(n-1)!} \cdot \frac{1}{z^{n-1}} + \dots + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{z} + 1.$$

(7)

In this case, to find $\text{Res}(e^{1/z}; 0)$, only technique ① of the above will work. So, even though technique ① is harder than ② and ③, it is still useful as sometimes it is our only option. (especially if we are not even sure whether the pole has finite order or not!)



Remark: We have explained that, when a pole a has finite order n , then

$$\text{Res}(f; a) = \frac{(z-a)^n \cdot f(z)}{(n-1)!} \Big|_{z=a}$$

However, while substituting $z=a$ in $((z-a)^n \cdot f(z))^{n-1}$ is certainly possible in the power series expression of $(z-a)^n \cdot f(z)$, it is not necessarily possible

in the original formula for $(z-a)^n \cdot f(z)$.

ex: $f(z) = \frac{\cos z}{\sin z}$. This has a simple pole at 0 of order 1.

So, to find $\text{Res}(f; 0)$, we take $\frac{z \cdot \cos z}{\sin z}$,

(8)

differentiate 0 times (i.e. not at all), and then
 substitute $z=0$ in $\frac{z \cdot \cos z}{\sin z}$; or, actually, by the proof!
 in its power series expression around 0,
 which, unlike $\frac{z \cdot \cos z}{\sin z}$, is defined at 0.

Notice that substituting $z=0$ in the power series is just like taking the $\lim_{z \rightarrow 0} \frac{z \cdot \cos z}{\sin z}$; so this is what substitution really means in this case.

This is not a surprise; since $\frac{z \cdot \cos z}{\sin z}$ can be written as a power series in a disc around 0, this power series also has to converge at 0, as any power series converges inside a whole disc.
 So, the power series is the holomorphic extension of $\frac{z \cdot \cos z}{\sin z}$ to the whole disc, including 0, not defined at 0.

$$\text{So, } g(z) = \begin{cases} \frac{z \cos z}{\sin z}, & z \neq 0 \text{ in disc} \\ a_0, & z=0 \end{cases} \text{ is holomorphic,}$$

so it is continuous, so $\lim_{z \rightarrow 0} g(z) = g(0)$,

$$\text{i.e. } \lim_{z \rightarrow 0} \frac{z \cos z}{\sin z} = a_0 = \text{the value of the power series for } z=0.$$

(3)

The above remark leads to the following technique:

IV If a is a pole of f of finite order, equal to (or at least) m , then

$$\text{Res}(f; a) = \frac{1}{(m-1)!} \cdot \lim_{z \rightarrow a} ((z-a)^m \cdot f(z))^{(m-1)}$$

(which of course equals $\frac{(z-a)^m f(z)}{(m-1)!} \Big|_{z=a}$)

when the limit exists).

IV for simple poles: Technique **IV** is of order 1

particularly useful when a is a pole of order 1.

Then, it translates to

$$\text{Res}(f; a) = \lim_{\substack{z \rightarrow a \\ \text{simple pole}}} (z-a) \cdot f(z)$$

for $f(z) = \frac{1}{z-a}$, $= 1$, $(z-a) \cdot f(z)$
so we can substitute $z=a$ without taking a limit. But not always the case.

this is the case, for instance, with $\text{Res}(f; 0)$ when $f(z) = \frac{\cos z}{\sin z}$. 0 is a simple pole,

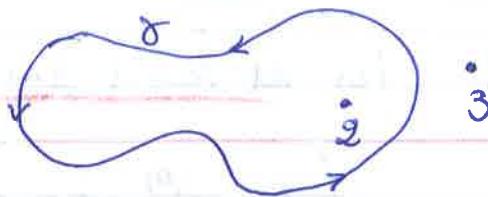
$$\text{so } \text{Res}(f; 0) = \lim_{z \rightarrow 0} (z-0) \cdot \frac{\cos z}{\sin z} = \lim_{z \rightarrow 0} \left(\frac{z}{\sin z} \cdot \cos z \right) =$$

$$= \lim_{z \rightarrow 0} \frac{z}{\sin z} \cdot \lim_{z \rightarrow 0} \cos z = 1 \cdot \cos 0 = 1.$$

(10)

Examples:Preliminary:

$$\bullet \int_{\gamma} \frac{1}{(z-2)^2} dz = \\ = \text{Res}\left(\frac{1}{(z-2)^2}; 2\right) = 0.$$



$$\bullet \int_{\gamma} \left(\frac{1}{(z-2)^3} + \frac{1}{(z-2)^4} + \underbrace{\frac{(z-1)}{(z-2)^4}}_{\text{residue}} \right) dz = 0.$$

$$\bullet \int_{\gamma} \left(\frac{10}{(z-3)^5} + \frac{99}{z-2} + 5 \right) dz = 99 \cdot 2\pi i.$$

holomorphic
around 2, so
power series centered at 2
(with only non-negative powers)
⇒ the residue is 99

$$\bullet \int_{\gamma} \left(f(z) + \frac{11}{z-2} \right) dz = 11 \cdot 2\pi i$$

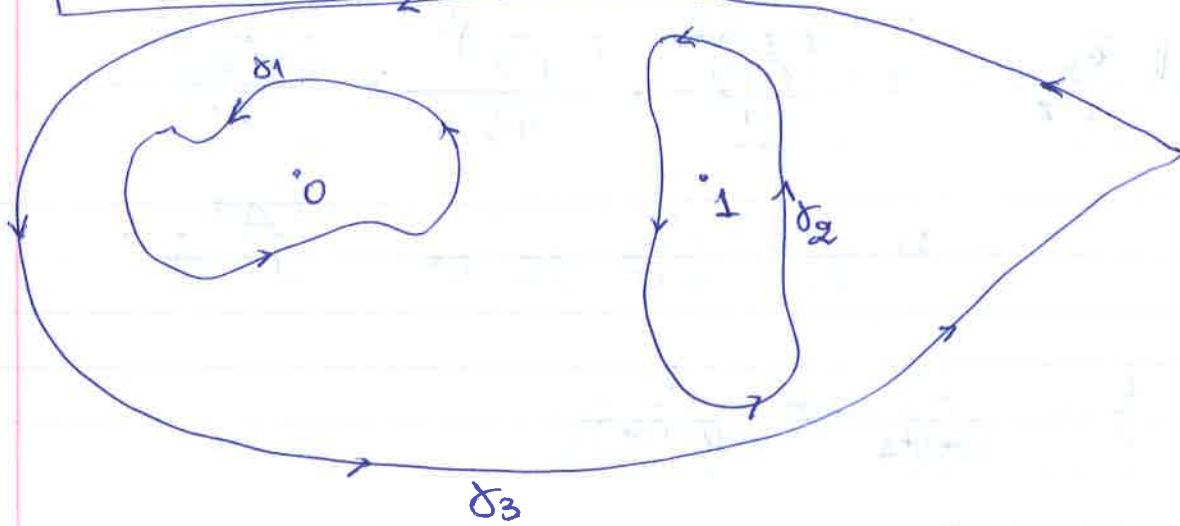
holomorphic
around 2

$$\bullet \int_{\gamma} e^{\frac{1}{z-2}} dz = \int_{\gamma} \left(1 + \frac{1}{z-2} + \frac{1}{2!} \cdot \frac{1}{(z-2)^2} + \frac{1}{3!} \cdot \frac{1}{(z-2)^3} + \dots \right) dz = \\ = 1 \cdot 2\pi i = 2\pi i.$$

(i)

$$f(z) = \frac{\cosh \frac{1}{1-z}}{z}$$

→ poles 0, 1



We know: $\int_{\delta_1} f(z) dz = 2\pi i \cdot \text{Res}(f; 0),$

$$\int_{\delta_2} f(z) dz = 2\pi i \cdot \text{Res}(f; 1),$$

$$\int_{\delta_3} f(z) dz = 2\pi i \cdot (\text{Res}(f; 0) + \text{Res}(f; 1)). \text{ And:}$$

→ 0 is a root of order 1 of the denominator,

while $\cosh \frac{1}{1-0} = \cosh 1 \neq 0.$

So, 0 is a pole of order 1. So,

$$\begin{aligned} \text{Res}(f; 0) &= \lim_{z \rightarrow 0} z \cdot f(z) = \lim_{z \rightarrow 0} z \cdot \frac{\cosh \frac{1}{1-z}}{z} = \\ &= \lim_{z \rightarrow 0} \cosh \frac{1}{1-z} = \cosh 1. \end{aligned}$$

(12)

$$\cosh w = \frac{e^w + e^{-w}}{2} = 1 + \frac{w^2}{2!} + \frac{w^4}{4!} + \frac{w^6}{6!} + \dots$$

$$\Rightarrow \cosh \frac{1}{z-1} = 1 + \frac{\left(\frac{1}{z-1}\right)^2}{2!} + \frac{\left(\frac{1}{z-1}\right)^4}{4!} + \frac{\left(\frac{1}{z-1}\right)^6}{6!} + \dots$$

$$= 1 + \frac{1}{2!} \cdot \frac{1}{(z-1)^2} + \frac{1}{4!} \frac{1}{(z-1)^4} + \frac{1}{6!} \frac{1}{(z-1)^6} + \dots \quad (*)_1$$

and $\frac{1}{z} = \frac{1}{(z-1)+1} = \frac{1}{1-(z-1)} =$

$$= (-z+1)^0 + (-z+1)^1 + (-z+1)^2 + \dots =$$

$$= 1 - z + (z-1)^2 - (z-1)^3 + (z-1)^4 - \dots \quad (*)_2$$

So, $\frac{\cosh \frac{1}{z-1}}{z}$ is the product of the series $(*)_1$ and $(*)_2$,

which is a new series with coefficient of $\frac{1}{z-1}$

coming from multiplying

$$\frac{1}{2!} \cdot \frac{1}{(z-1)^2} \quad \text{with } -(z-1),$$

$$\frac{1}{4!} \cdot \frac{1}{(z-1)^4} \quad \text{with } -(z-1)^3,$$

$$\frac{1}{6!} \cdot \frac{1}{(z-1)^6} \quad \text{with } -(z-1)^5, \text{ etc.}$$

So, the coefficient of $\frac{1}{z-1}$ is

(13)

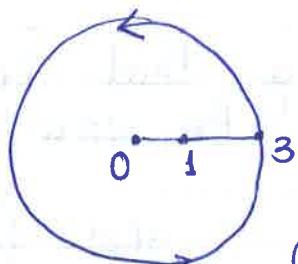
$$-\frac{1}{2!} - \frac{1}{4!} - \frac{1}{6!} - \dots = -\left(\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots\right) = \\ = -(\cosh 1 - 1) = 1 - \cosh 1.$$

I.e. : $\text{Res}(f; 1) = 1 - \cosh 1.$

(ii) $f(z) = \frac{z \operatorname{Re} z}{(z-1)^2} \cdot \int_{C(0,3)} f(z) dz = ?$

We cannot use the

residue theorem, because $C(0,3) = \{3e^{it} : t \in [0, 2\pi]\}.$



f is not holomorphic anywhere (i.e., its singularities are not isolated).

In this case, we have two options:

Option 1: Use definition of $\int_C f(z) dz :$

$$\int_{C(0,3)} f(z) dz = \int_0^{2\pi} f(3e^{it}) \cdot 3ie^{it} dt \stackrel{\operatorname{Re}(3e^{it}) = 3\cos t}{=} \\ = \int_0^{2\pi} f(3e^{it}) \cdot 3ie^{it} dt,$$

$$= \int_0^{2\pi} \frac{3e^{it} \cdot 3\cos t}{(3e^{it}-1)^2} \cdot 3ie^{it} dt, \text{ and calculate this.}$$

But this is hard!

Option 2 is much more clever:

(14)

Option 2: Notice that, if $f(z) = g(z) \ \forall z \in \gamma^*$,
then $\int_{\gamma} f(z) dz = \int_{\gamma} g(z) dz$.

So, all we need to do is :

(i) find g that has only isolated singularities inside the area γ surrounds,

s.t. $g(z) = f(z) \ \forall z \in \gamma^*$ (don't care about outside the curve)

(ii) $\int_{\gamma} f(z) dz = \sum_{\text{a the}} \text{Res}(g; a)$
 singularities
 $\text{of } g$ surrounded
 $\text{by } \gamma$.

(we can apply the residue theorem for g !).

Let's apply this to our f :

$$\text{On } C(0,3), \underbrace{|z|}_{\parallel}^g = 3 \Leftrightarrow \bar{z} = \frac{3}{z}$$

$$\text{So, } \operatorname{Re} z = \frac{z + \bar{z}}{2} = \frac{z + \frac{3}{z}}{2} \quad \text{for } z \text{ on } C(0,3)$$

(only, but we
don't care about other z).

(15)

$$\text{So, } \int_{C(0,3)} \frac{z \operatorname{Re} z}{(z-1)^2} dz = \int_{C(0,3)} \frac{z \cdot \left(z + \frac{9}{z}\right)}{2(z-1)^2} dz = \int_{C(0,3)} \left(\frac{z^2 + 9}{2(z-1)^2} \right) dz =$$

only singularity
surrounded by $C(0,3)$
is 1 ! So, we
can use residue theorem.

$$= 2\pi i \cdot \operatorname{Res} \left(\frac{z^2 + 9}{2(z-1)^2}; 1 \right).$$

Now: 1 is a root of order 2 of the denominator,
and $1^2 + 9 = 10 \neq 0$.

So, 1 is a pole of order 2 .

To find the residue, we can use

$$\begin{aligned} \operatorname{Res} \left(\frac{z^2 + 9}{2(z-1)^2}; 1 \right) &= \left((z-1)^2 \cdot \frac{z^2 + 9}{2(z-1)^2} \right)''(1) = \\ &= \left(\frac{z^2 + 9}{2} \right)''(1) = \left(\frac{2z}{2} \right)'(1) = 1. \end{aligned}$$

Or we can calculate the Laurent series:

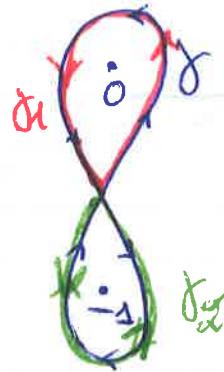
$$z^2 + 9 = (z-1)^2 + 2z - 1 + 9 = (z-1)^2 + 2z + 8 = (z-1)^2 + 2(z-1) + 10,$$

$$\text{so } \frac{z^2 + 9}{2(z-1)^2} = \frac{1}{2} + \frac{2}{2(z-1)} + \frac{10}{2(z-1)^2} = \frac{1}{2} + \frac{1}{z-1} + \frac{5}{(z-1)^2},$$

$$\text{so } \operatorname{Res} \left(\frac{z^2 + 9}{2(z-1)^2}; 1 \right) = 1.$$

$$(iii) \quad f(z) = \frac{\sin z}{z(z+1)} \quad \int \limits_{\gamma} f(z) dz = ?$$

(16)



We know that f only has two poles.

But the problem here is that γ is not anti clockwise (not clockwise either), so we cannot apply the residue theorem. To deal with this, split in two parts, the red anticlockwise γ_1 and the green anticlockwise γ_2 , and notice that

$$\int \limits_{\gamma} f(z) dz = \int \limits_{\gamma_1} f(z) dz - \int \limits_{\gamma_2} f(z) dz$$

\parallel \parallel

$\text{anti. Res}(f; 0) \quad \text{anti. Res}(f; -1).$

Now: 0 is not really a pole:

$$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots$$

a convergent power series $\forall z \neq 0$ (so also for $z=0$, as

(17)

power series converge in whole discs).

And $\frac{1}{z+1}$ is holomorphic around 0, so $\frac{1}{z+1} \cdot \frac{\sin z}{z}$ is a power series around 0, so
 $\text{Res}\left(\frac{\sin z}{z(z+1)}; 0\right) = 0$.

And: -1 is a root of the denominator of order 1,

and $\sin(-1) \neq 0$, so

-1 is a pole of order 1 (a simple pole),

$$\text{thus } \text{Res}\left(\frac{\sin z}{z(z+1)}; -1\right) = \lim_{z \rightarrow -1} (z + 1) \frac{\sin z}{z(z+1)} =$$


$$= \lim_{z \rightarrow -1} \frac{\sin z}{z} = \frac{\sin(-1)}{-1} = -\sin(-1).$$

So, $\int \frac{\sin z}{z(z+1)} dz = 2\pi i \cdot (-\sin(-1)) = 2\pi i \cdot \sin 1$.

①

Lecture 13:

→ Applications of complex analysis:

→ Evaluating definite integrals in \mathbb{R} :

Usual cases:

① $\int_a^b f(x) dx$, $a, b \in \mathbb{R}$.

② $\int_{-\infty}^{+\infty} f(x) dx$ when f is defined everywhere in \mathbb{R} .

③ $\int_0^{+\infty} f(x) dx$ or $\int_{-\infty}^0 f(x) dx$, when f has singularities (finitely many) in \mathbb{R} .

Let's see this in more detail:

① $\int_a^b f(x) dx$, $a, b \in \mathbb{R}$

We rename x by t , so that the integral we have, $\int_a^b f(t) dt$,

reminds us a little of the definition of curve integrals.

Idea: Try to bring $\int_a^b f(t) dt$ in the form

$$\int_a^b g(f(t)) d(g(t)), \text{ for some complex function } g: J^* \rightarrow \mathbb{C}$$

i.e., we want to write $f(t) dt = g(f(t)) d(g(t))$, for some g .

If we can do this, then $\int_a^b f(x) dx = \int g(z) dz$, and

(2)

thus we can use what we've learnt so far.

Method: • Guess the appropriate $\gamma(t)$, $t \in [a, b]$.

- $d(\gamma(t)) = \gamma'(t) dt$, so replace dt by $\frac{d(\gamma(t))}{\gamma'(t)}$
- We are left with $\int_a^b \frac{f(t)}{\gamma'(t)} d(\gamma(t))$;
try to write $\frac{f(t)}{\gamma'(t)}$ as $g(\gamma(t))$

Then, $\int_a^b f(t) dt = \int_a^b g(\gamma(t)) d(\gamma(t)) = \int_{\gamma} g(z) dz$

→ **Common use:**

When $g(t)$ involves $\boxed{\sin}$ and $\boxed{\cos}$,

and $[a, b] = [0, 2\pi]$

ex: $\int_0^{2\pi} \frac{dt}{3 + \cos t}$

Usual method: Write $\sin t = \frac{e^{it} - e^{-it}}{2i}$, $\cos t = \frac{e^{it} + e^{-it}}{2}$,
and $d(e^{it}) = ie^{it} dt \Rightarrow dt = \frac{d(e^{it})}{ie^{it}}$.

Then, the integral will be transformed in an integral on the circle!

(3)

for the example: $\int_0^{2\pi} \frac{dt}{3+\cos t}$

Write $\cos t = \frac{e^{it} + e^{-it}}{2}$, $d(e^{it}) = ie^{it} dt \Rightarrow dt = \frac{d(e^{it})}{ie^{it}}$,

so $\int_0^{2\pi} \frac{dt}{3+\cos t} = \int_0^{2\pi} \frac{1}{3 + \frac{e^{it} + e^{-it}}{2}} \cdot \frac{1}{ie^{it}} d(e^{it}) =$

$$= \int_{C(0,1)} \frac{1}{3 + \frac{z + \frac{1}{z}}{2}} \cdot \frac{1}{iz} dz =$$

$$= \int_{C(0,1)} \frac{2}{6 + z + \frac{1}{z}} \cdot \frac{1}{iz} dz = \frac{2}{i} \cdot \int_{C(0,1)} \frac{1}{6z + z^2 + 1} dz$$

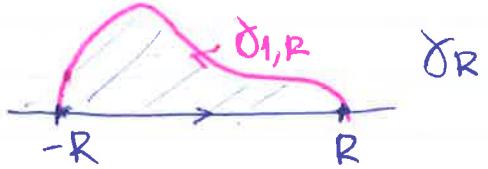
Now, as usual: find roots, they are the poles, apply residue theorem.

$\boxed{2i \int_{-\infty}^{+\infty} f(t) dt}$, f has no singularities on \mathbb{R} .

We have: $\int_{-\infty}^{+\infty} f(t) dt = \lim_{R \rightarrow \infty} \int_{-R}^R f(t) dt$.

Idea: Extend f further from the real line, to create a closed curve γ_R like this:

Then:



(4)

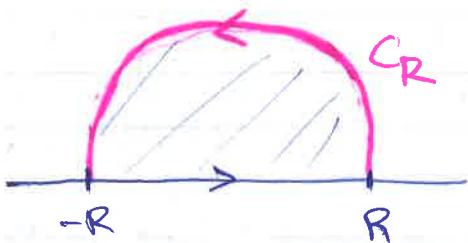
$$\int_{-R}^R f(t) dt + \int_{\gamma_{1,R}} f(z) dz = 2\pi i \operatorname{Res}(f; \alpha)$$

a the
singularities
of f (the extended f)
surrounded by γ_R

$= I_R$, something
we can
calculate.

Hopefully, as R goes to ∞ , we can say
something for the limits of I_R and $\int_{\gamma_{1,R}} f(z) dz$.

→ **Common method:** Extend f to the upper half-plane
(or lower!)



Apply residue theorem
in shaded area, to get

$$\int_{-R}^R f(t) dt + \int_{C_R} f(z) dz = 2\pi i \cdot \sum \operatorname{Res}(f; \alpha)$$

a the singularities of f
in the shaded area

as $R \rightarrow \infty$,
this becomes

$2\pi i \sum \operatorname{Res}(f; \alpha)$
a all the
singularities of f
in the upper half plane!

(5)

So, the goal is just to find $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz$! Once we do
we have

$$\int_{-\infty}^{+\infty} f(t) dt = 2\pi i \sum \text{Res}(f; a) - \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz.$$

a the singularities of f in the upper half plane

↓ careful if you extend to lower half plane

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz.$$

How do we deal with something like this?

Tool 1: Triangle inequality: For any $h: \mathbb{R} \rightarrow \mathbb{C}$,

to show that

$$\int_{C_R} f(z) dz = 0; \text{ otherwise won't work.}$$

$$\left| \int_a^b h(t) dt \right| \leq \int_a^b |h(t)| dt$$

⚠ make sure $t \in \mathbb{R}$ to apply this!

So, when $f: \gamma^* \rightarrow \mathbb{C}$, where $\gamma: [a, b] \rightarrow \mathbb{C}$, we have:

$$\left| \int_{\gamma} f(z) dz \right| \leq (\text{max value of } |f| \text{ on } \gamma) \cdot (\text{length of } \gamma)$$

Proof: $\left| \int_{\gamma} f(z) dz \right| = \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \leq$

triangle inequality

$$\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| dt \leq$$

(6)

$$\leq \int_a^b \underbrace{\max_{t \in [a,b]} |f(\gamma(t))|}_{\text{a number independent of } t} \cdot |\gamma'(t)| dt =$$

$$= \underbrace{\max_{t \in [a,b]} |f(\gamma(t))|}_{\parallel} \cdot \underbrace{\int_a^b |\gamma'(t)| dt}_{\parallel \text{length of } \gamma}$$

$$\max_{z \in \gamma^*} |f(z)|$$

■

for us:

$$\left| \int_{C_R} f(z) dz \right| \leq \underbrace{\max_{z \in C_R} |f(z)|}_{\parallel} \cdot \underbrace{\text{length of } C_R}_{\frac{2\pi R}{2}} =$$

$$= \pi \cdot \underbrace{\max_{z \in C_R} |f(z)|}_{\parallel} \cdot R.$$

$$\max_{t \in [0,\pi]} |f(R e^{it})|$$

\downarrow
this form
may be more
helpful!

So, if we can show

that $\max_{z \in C_R} |f(z)| \leq \frac{\text{constant}}{R^{1+\theta}}$ $\theta > 0$

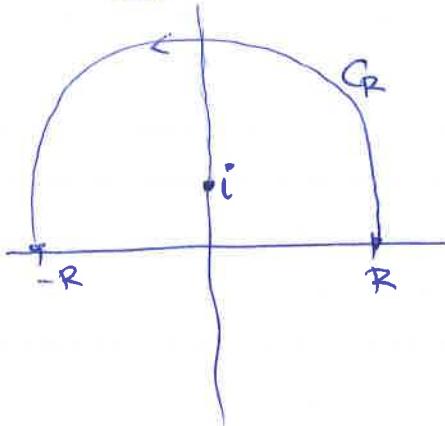
then $\left| \int_{C_R} f(z) dz \right| \leq$ for R large,

$$\leq (\text{constant}) \cdot \frac{1}{R^{\theta}} \xrightarrow[R \rightarrow \infty]{} 0,$$

$$\text{so } \int_{-\infty}^{+\infty} f(t) dt = 2\pi i \sum \text{Res}(f; \alpha)$$

at the singularities of f
in the upper half-plane

ex: $\int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt = \lim_{R \rightarrow +\infty} \int_{-R}^R \frac{1}{1+t^2} dt.$



$$\text{so, } \int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt = 2\pi i \cdot \text{Res}\left(\frac{1}{1+z^2}; i\right) - \lim_{R \rightarrow +\infty} \int_{C_R} \frac{1}{1+z^2} dz.$$

And $\left| \int_{C_R} \frac{1}{1+z^2} dz \right| \leq \underbrace{\max_{z \in C_R} \left| \frac{1}{1+z^2} \right|}_{\sim \frac{1}{R^2}} \cdot (\text{length of } C_R) \leq \underbrace{n \cdot R}_{\sim R}$

$\max_{t \in [0, n]} \left| \frac{1}{1+(Re^{it})^2} \right|$

$\sim \frac{1}{R^2}$ for R large, so $\leq \frac{\text{constant}}{R^2}$

$\leq (\text{constant}) \cdot \frac{R}{R^2} \xrightarrow[R \rightarrow +\infty]{} 0$
for large R

$\rightarrow \text{so: } \int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt =$
 $= 2\pi i \cdot \text{Res}\left(\frac{1}{1+z^2}; i\right).$

(8)

This tool deals with $\int_{-\infty}^{+\infty} \frac{P(t)}{Q(t)} dt$ polynomials

and $\int_{-\infty}^{+\infty} \frac{P(t)}{Q(t)} e^{iat} dt$, just a rotation for every t

when $\deg Q \geq \deg P + 2$ and $a > 0$.

for $a < 0$, we need to extend in lower half-plane.

Tool 2: Jordan's Lemma: $\int_0^\pi e^{-R\sin\theta} d\theta \leq \frac{\pi}{R}$

This tool manages to deal with $\int_{-\infty}^{+\infty} \frac{P(t)}{Q(t)} e^{iat} dt$

when $\deg Q \geq \deg P + 1$ and $a > 0$. See examples in textbook.

$$\left(\left| \int_{C_R} f(z) dz \right| \leq \int_0^\pi |f(Re^{it})| \cdot |e^{ia(Re^{it})}| \cdot |iRe^{it}| dt \leq \int_0^\pi e^{-R\sin t} dt \leq c \cdot \frac{1}{R} \xrightarrow{R \rightarrow \infty} 0 \right)$$

$\underbrace{|f(Re^{it})|}_{\leq \frac{1}{R}} \cdot |e^{ia(Re^{it})}| \cdot |iRe^{it}|$
 $\leq c \cdot \frac{1}{R} \cdot |e^{ia(Re^{it})}| \cdot |e^{-R\sin t}|$
 $\leq c \cdot \frac{1}{R} \cdot |e^{ia(Re^{it})}| \cdot e^{-R\sin t}$

We write $\int f(z) dz = \int_0^\pi f(Re^{it}) \cdot Re^{it} dt$. If it is clear that,

$$(Re^{it})' dt =$$

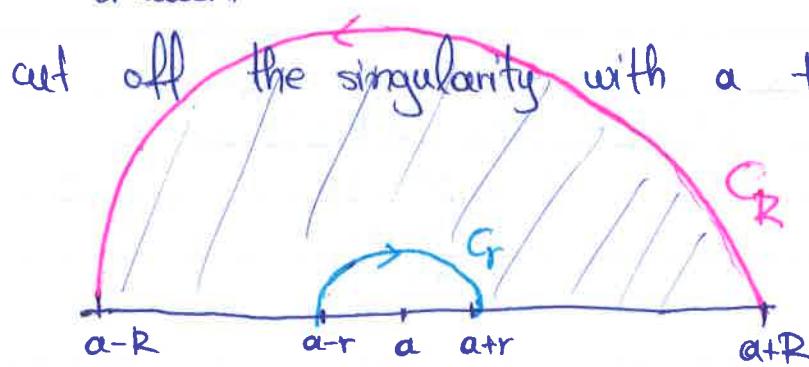
2 ii

$\int_{-\infty}^{+\infty} f(t) dt$, f has a singularity in \mathbb{R}
(or, anyway, finitely many singularities).

Suppose that f has a singularity at $a \in \mathbb{R}$. Then,

$$\int_{-\infty}^{+\infty} f(t) dt = \lim_{\substack{R \rightarrow +\infty \\ r \rightarrow 0}} \left(\int_{a-R}^{a-r} f(t) dt + \int_{a+r}^{a+R} f(t) dt \right),$$

and to find this limit we extend f as before in the upper half-plane, but this time we also or lower!



cut off the singularity with a tiny circle:
(We cut off because we can't apply the residue theorem if there are singularities on the boundary!)

(10)

By the residue theorem in the shadowed area,

$$\int_{a-R}^{a-r} f(t) dt + \int_{C_R} f(z) dz + \int_{a+r}^{a+R} f(t) dt + \int_{C_R} f(z) dz =$$

$$= 2\pi i \sum \text{Res}(f; a)$$

a the
sing. of f
in shadowed
area

$$\downarrow \text{as } R \rightarrow \infty$$

$$2\pi i \sum \text{Res}(f; a)$$

a the
singularities of
 f in upper half-plane
(not on \Re !)

So, all we need is to find $\lim_{r \rightarrow 0} \int_G f(z) dz = I_{\text{small}}$

$$\text{and } \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = I_{\text{large}}$$

then, $\int_{-\infty}^{+\infty} f(t) dt = 2\pi i \sum \text{Res}(f; a) - I_{\text{large}} + I_{\text{small}}$

a the sing.
of f in upper
half plane

(11)

To find I_{large} and I_{small} , we can use the two tools above. However, we generally don't expect I_{small} to be 0, so in that case those tools won't work. Instead, we use continuity of the integrand at 0, as $r \rightarrow 0$:

Tool: Continuity:

$$\int_C f(z) dz = \int_0^\pi f(re^{it}) d(re^{it}) = \int_0^\pi f(re^{it}) \cdot i \cdot re^{it} dt.$$

If we have that $f(re^{it}) \cdot ire^{it} \xrightarrow[r \rightarrow 0]{} c_0$,

for some complex number c_0 , then

$$\int_C f(z) \xrightarrow[r \rightarrow 0]{} \int_0^\pi c_0 dt = c_0 \cdot \pi.$$

ex: $\int_{-\infty}^{+\infty} \frac{e^{it}}{t} dt$ (see textbook).

For this, we have: $\int_C \frac{e^{iz}}{z} dz = \int_0^\pi \frac{e^{i(re^{it})}}{re^{it}} d(re^{it}) =$
 $= \int_0^\pi \frac{e^{i(re^{it})}}{re^{it}} \cdot i \cdot re^{it} dt \xrightarrow[r \rightarrow 0]{} \int_0^\pi e^{i(0 \cdot e^{it})} i dt = \int_0^\pi i dt = i \cdot \pi.$

keep this
as a whole

(12)

(3) $\int_0^{+\infty} f(t) dt$, or $\int_{-\infty}^0 f(t) dt$, where f is even

(i.e., $f(-t) = f(t) \forall t \in \mathbb{R}$).

Then, these integrals equal $\frac{1}{2} \int_{-\infty}^{+\infty} f(t) dt$,

which we can try to calculate as in (2).



Let's go back to $\int_{-\infty}^{+\infty} \frac{e^{it}}{t} dt$. We see

eventually that $\int_{-\infty}^{+\infty} \frac{e^{it}}{t} dt = \pi i$

$$\underbrace{\int_{-\infty}^{+\infty} \frac{e^{it}}{t} dt}_{\pi i} = \int_{-\infty}^{+\infty} \frac{\sin t}{t} dt + i \int_{-\infty}^{+\infty} \frac{\cos t}{t} dt.$$

So, $\int_{-\infty}^{+\infty} \frac{\sin t}{t} dt = \text{Im}(\pi i) = \pi$

→ Thus, if we were asked to find $\int_{-\infty}^{+\infty} f(\sin t) dt$,

we could

find $\int_{-\infty}^{+\infty} f(e^{it}) dt$,

and then check if $\int_{-\infty}^{+\infty} f(\sin t) dt$ is

its real (or imaginary) part.

Not true

→ Notice in general! that $\int_{-\infty}^{+\infty} \frac{\cos t}{t} dt = \text{Re} \left(\int_{-\infty}^{+\infty} \frac{e^{it}}{t} dt \right) = 0$, by the above.

(13)

But close to 0, cost ~ 1 , so $\frac{\text{cost}}{t} \sim \frac{1}{t}$, which is not integrable!

Also, $\frac{\text{cost}}{t}$ is an odd function, so: $\int_{-\infty}^{+\infty} \frac{\text{cost}}{t} dt = +\infty \Rightarrow \int_0^{\infty} \frac{\text{cost}}{t} dt = -\infty$; how did we add those two to get 0?

Nothing: the technique we followed actually found

$$\lim_{\substack{R \rightarrow +\infty \\ \text{and } r \rightarrow 0}} \left(\int_{-R}^{-r} \frac{\text{cost}}{t} dt + \int_r^R \frac{\text{cost}}{t} dt \right);$$

$$\text{not } \int_{-\infty}^0 \frac{\text{cost}}{t} dx, \text{ nor } \int_{-\infty}^{+\infty} \frac{\text{cost}}{t} dt.$$

The only "mistake" was our initiative to naively

call the above limit $\int_{-\infty}^{+\infty} \frac{\text{cost}}{t} dt$ In situations

like this, where $\int_{-\infty}^0 f(t) dt = -\infty$ and $\int_0^{+\infty} f(t) dt = +\infty$,

we prefer to denote

$$\lim_{\substack{R \rightarrow +\infty \\ r \rightarrow 0}} \left(\int_{-R}^{-r} f(t) dt + \int_r^R f(t) dt \right) \text{ by p.v. } \int_{-\infty}^{+\infty} f(t) dt$$

↓
principal value

The above discussion leads to the conclusion that we can find residue theory useful when:

→ **Evaluating principal values**

(See examples in book).

(14)

→ Finding the inverse Laplace transform:

We will not dwell on this now; we will return when we see the Laplace transform in detail. The idea is that the Laplace transform sends a function $f: \mathbb{R} \rightarrow \mathbb{R}$ to another function, $\mathcal{L}(f)$. The question is, if we know $\mathcal{L}(f)$, how can we find f ??

One can show that f can be expressed as an integral of $\mathcal{L}(f)$ (see p. 697 in book)

and then, using residue theory, f can be found.

→ Hilbert transforms:

Let $u: \mathbb{R} \rightarrow \mathbb{R}$,

that is the real part of the boundary

values (on the real line) of some $f: \text{upper half-plane} \rightarrow \mathbb{C}$

$$f(z) = u(z) + i v(z)$$

For u with not too bad behaviour, such an extension to a harmonic function u on the whole upper half-plane exists; just convolve u with the Poisson kernel, read elsewhere if you are interested.

If we know we can extend u to a harmonic function defined on the whole upper half-plane, how do we find $v: \mathbb{R} \rightarrow \mathbb{R}$, the restriction of the harmonic conjugate of u to \mathbb{R} ? The answer is: via the Hilbert transform.

→ Def.: Let $f: \mathbb{R} \rightarrow \mathbb{R}$. We define the Hilbert transform

$$\underbrace{H(f)}_{\text{another real function}}: \mathbb{R} \rightarrow \mathbb{R} \quad t \mapsto \frac{1}{\pi} \operatorname{p.v.} \int_{-\infty}^{+\infty} \frac{f(x)}{x-t} dx$$

$$\lim_{\substack{R \rightarrow +\infty \\ r \rightarrow 0}} \left(\int_{t-R}^{t-r} \frac{f(x)}{x-t} dx + \int_{t+r}^{t+R} \frac{f(x)}{x-t} dx \right)$$

Via residue theory (p. 698) we can show that, for u as above, the desired harmonic conjugate restricted to \mathbb{R} is $v = -\overline{H(u)}$!

(16)

And also, $u = \mathcal{H}(v)$.

(Notice that $\mathcal{H}(\mathcal{H}(u)) = \mathcal{H}(-v) = -\mathcal{H}(v) = -u$,

i.e. the Hilbert transform of the Hilbert transform
is $-$ (the original function)).

Hilbert transforms are very useful in signal processing.

In particular, if $u: \mathbb{R} \rightarrow \mathbb{R}$ is a periodic signal,

we want to understand its frequencies to understand
 u itself.

(this really means that understanding the
Fourier series of u implies things for u ; we
will see more on Fourier series in the future).

But the negative frequencies are just the conjugates
of the positive. So, we just need to understand
the positive frequencies. And, in fact,

$u(t) + i\mathcal{H}(u)(t)$
the analytic representation of the signal.

has only the positive frequencies
of u as frequencies, so it preserves
the information we want.